Penetration through a Slot in a Conducting Plane Backed by a Conducting Channel, Part I: TE Case

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Abstract

The penetration of the electromagnetic field through an infinite slot in a conducting plane backed by a channel is examined. The total TE-to-slot-axis field in the interior of the channel and above the conducting plane is determined by two independent integral equation methods: 1) a coupled integral equation method which can be used to determine the field in the case of a channel with an arbitrarily-shaped cross-section and 2) a single integral equation method which involves a Green’s function specific to the shape of the backing channel. Data from the two methods are presented for the equivalent currents obtained from the integral equations, for the field near the channel-backed slot, and for the actual currents present on the conducting surfaces.

1. Introduction

Computation of the electromagnetic field that penetrates a conducting surface through an aperture backed by a cavity or channel has been the focus of many studies as one learns from a brief review of the references of [1]. The present interest is in integral equation methods for determining the field that passes through an infinite slot of uniform width in a planar conducting surface into the interior of a channel behind the slotted screen, caused by an incident field TE to, and invariant along, the slot axis. Asvestas and Kleinman [2] introduce a coupled integral equation procedure that was later altered by Wood and Wood [3] to accommodate a 2-dimensional arbitrarily-shaped trough filled with a material other than that of the upper half-space. Whites, Michielssen, and Mittra [4] solve for the fields via a method of moments technique involving the impedance boundary condition. Other authors employ hybrid methods comprising finite element and method of moments techniques to allow for an inhomogeneous material in the cavity ([5], [6], [7]). Asymptotic and eigenvalue expansion methods have been
used to compute the field for the specific case of a ground plane trough of hemicylindrical cross-section ([8], [9], [10], [11]). Eigenvalue expansion methods are also used to compute the field for the specific elliptically-shaped channel backing in [12] and [13] and a wedge loaded by a slot in [14].

In this investigation, we present two independent integral equation methods for computing the field in a filled channel behind an infinite slot in a conducting screen illuminated by a field TE to the slot axis. For each of the canonical channel cross-sections, hemicylindrical, sectorial, and rectangular, a single integral equation is derived and results obtained from the solutions of these equations are compared with those from a coupled integral equation valid for an arbitrarily-shaped channel backing. Unlike the case of a trough in a ground plane, the width of the slot can be less than the width of the channel. Solutions obtained by two independent methods allow one to investigate validity and acquire confidence in accuracy of data determined.

The total fields near, and the currents on, the structures illustrated in Figure 1 are determined. Each structure is an infinite conducting plane with an infinite slot of uniform width, backed by a conducting channel filled, in general, with material different from that in the half-space in front of the plane. The excitation in the open half-space is either a time-harmonic plane wave whose magnetic field is parallel to the slot axis and invariant with respect to displacement along the slot axis or a time-harmonic magnetic line current parallel to the slot axis. The techniques developed are valid for the line current inside the channel. The signals vary in time according to $e^{j\omega t}$ which is suppressed. A source that is TE-polarized with respect to the z-axis is considered since, in general, this polarization leads to stronger penetration of fields through the aperture than one expects from an excitation that is transverse magnetic to the slot axis. Data obtained from a coupled integral equation method devised for finding fields and currents for an arbitrarily shaped
channel filled with a general material characterized by \((\mu_z, \varepsilon_z)\), such as that illustrated in Figure 2, are compared with those of a single integral equation method involving a Green’s function specific to the channel shapes of Figure 1.

2. Integral Equation Formulations

A derivation of coupled integral equations for the structure and excitation of Figure 2 is presented as is the formulation of a single integral equation containing a Green’s function specific to the channel shapes of Figure 1. From the solutions of the integral equations, the total field in the half-space and in the backing channel is determined as are the actual currents on the conducting surfaces. In the formulation of the integral equations, the aperture (slot) of the structure is shorted and a \(z\)-directed magnetic surface current \(M_z(x)\) is placed on the exterior surface \((y > 0)\) of the shorted slot, \(x \in (x_1, x_2)\), while a surface current \(-M_z(x)\) is placed on the interior surface \((y < 0)\) (Figure 3), which causes the total \(x\)-directed electric field \(E_x(x, y)\) to satisfy the condition, \(\lim_{y \to 0+} E_x^{(1)}(x, y) = \lim_{y \to 0-} E_x^{(2)}(x, y), x \in (x_1, x_2)\), where the superscript \(\{1\}\) identifies a quantity peculiar to the half-space (channel interior) ([15],[16],[17]). The coupled integral equation method includes an integral equation which forces the component of the electric field tangential to the channel backing to be zero at the backing, i.e., \(\mathbf{E}^{(2)} \cdot \hat{t} = 0\), where \(\hat{t}\) is defined as the unit vector tangential to the surface of the backing channel, and another which enforces continuity of the \(z\)-directed component of the magnetic field through the aperture. The specific Green’s function method requires only a single integral equation and incorporates the PEC condition at the channel backing into the Green’s function. The model of
Figure 3 with shorted slot, equivalent magnetic currents, and original source is equivalent to the original structure and source for both interior and exterior regions.

Field in Exterior Region

A model whose field is equivalent in the exterior region ($y>0$) to that in the exterior region of the structure in Figure 3 is introduced in Figure 4a. The total $z$-directed magnetic field in the exterior region is that due to the magnetic current and its image plus the magnetic field due to the original source and its image. One notes that the equivalent model contains no conductors or material discontinuities, and so free space Green's functions are used to compute field components from currents. The total magnetic field in the exterior region can be written as

$$H_z^{(1)}(x,y) = -\frac{k_1 K}{4\eta_1} H_0^{(2)}(k_1 R_0) - \frac{k_1 K}{4\eta_1} H_0^{(2)}(k_1 R'_i) - \frac{k_1}{4\eta_1} \int_{x_1}^{x_2} 2M_z(x') H_0^{(2)}(k_1 R_0) dx'$$

(1)

where

$$R_0 = \sqrt{(x-x')^2 + y^2}$$
$$R_s = \sqrt{(x-x_s)^2 + (y-y_s)^2}$$
$$R'_i = \sqrt{(x-x'_s)^2 + (y+y'_s)^2}$$

(2)

where $k_1 = \omega \sqrt{\mu_1 \varepsilon_1}$ is the wave number in region 1, $\eta_1 = \sqrt{\mu_1 / \varepsilon_1}$ is the intrinsic impedance of the material in region 1, and where $H_\nu^{(2)}(\bullet)$ is the $\nu$th order Hankel function of the second kind.

Coupled Integral Equations

A model equivalent in the interior region to the structure in Figure 3 is introduced in Figure 4b. The model includes the $z$-directed magnetic current $-M_z$ on the interior side of the shorted aperture and its image $-M_z$. The channel backing is imaged about $y=0$, and this imaged backing (contour $C'$ of Figure 4b) is replaced with an equivalent electric current $J_i$ in the direction transverse to the $z$-axis and tangential to $C'$.

In the model of Figure 4b the electric
field $E^{(2)}_x$ is zero for $y = 0$ due to the use of image theory. Coupled integral equations enforce
1) continuity of tangential magnetic field in the aperture and 2) the requirement that the
tangential electric field in the channel be zero on the interior surface of the channel wall. The
latter requirement is enforced by an equation that is reminiscent of the magnetic field integral
equation. Continuity of the $z$-directed magnetic field through the aperture is enforced by

$$\mathcal{H}_{i,M}^{(1)}[2M_z;x,0] - \mathcal{H}_{i,M}^{(2)}[-2M_z;x,0] - \mathcal{H}_{i,j,1}^{(2)}[J_i;x,0] = -H_{z}^{sc}(x,0), \ x \in (x_1,x_2), \ (3)$$

in which

$$\mathcal{H}_{i,M}^{(n)}[M_z;x,y] = -\frac{k_n}{2\eta_n} \int_{x_1}^{x_2} M_z(x') H_0^{(2)}(k_n R_0) \, dx', \quad (4)$$

$$\mathcal{H}_{i,j,1}^{(2)}[J_i;x,y] = -\frac{k_n}{2\eta} \int_{c'} J_i(x',y') \cos(\theta') H_1^{(2)}(k_n R) \, dl', \quad (5)$$

with

$$\cos(\theta') = \hat{u} \cdot \hat{n'}, \ \hat{u} = \frac{\rho - \rho'}{|\rho - \rho'|}, \ \hat{n'} = \hat{e} \times \hat{z}$$

$$R = |\rho - \rho'| = \sqrt{\rho^2 + \rho'^2 - 2\rho \rho' \cos(\phi - \phi')} = \sqrt{(x - x')^2 + (y - y')^2} \quad (6)$$

and

$$H_{z}^{sc}(x,y) = -\frac{k_1 K}{4\eta_1} H_0^{(2)}(k_1 R_0) - \frac{k_1 K}{4\eta_1} H_0^{(2)}(k_1 R_1), \quad (7)$$

where $\ell$ is defined as the displacement along contour $C'$. $H_{z}^{sc}$ is the short-circuit magnetic
field due to the magnetic line current radiating in the presence of the shorted slot. The integral
equation that forces the tangential electric field to be zero on the interior surface of the channel
backing is

$$\mathcal{H}_{i,M}^{(2)}[-2M_z;x,y] + \mathcal{H}_{i,j,2}^{(2)}[J_i;x,y] = 0, \ (x,y) \in C'. \quad (8)$$
where \( \mathcal{H}_{s,M}^{(2)} [M_s; x, y] \) is given in (4) and

\[
\mathcal{H}_{s,J,2}^{(2)} [J_s; x, y] = -\frac{J_s (x, y)}{2} - \frac{k_2}{4j} \int_{c'} J_e (x', y') \cos \left( \theta' \right) H_{1,2}^{(2)} (k_2 R) dl'
\]

(9)

where \( \cos (\theta') \) is defined in (6). The coupled integral equations (3) and (8) are solved numerically for the magnetic and electric currents and these currents are used to compute the total field present in both regions of the original structure of Figure 2. This method is valid for an arbitrarily-shaped channel filled with any homogeneous, isotropic material. The field computed from its solution is used to validate the field computed from the solution of the specific Green’s function integral equation discussed in the following sections.

Specific Green’s Function Method

A single integral equation can be derived for cases that the channel cross-section is one of several specific shapes. This integral equation is constructed in such a way that it enforces both continuity of z-directed magnetic field as the aperture is approached in the equivalent models for the two regions and the boundary condition \( \mathbf{E} \times \mathbf{n} = 0 \) on the interior surface of the channel wall [18]. A single integral equation that accounts for this boundary condition can be derived only for channels of specific shapes, not for an arbitrary shape like that in Figure 2. When the channel shape is, indeed, amenable to the single integral equation characterization, numerical solution efficiency is enhanced. For this integral equation the exterior region equivalent model remains as shown in Figure 4a and the interior region models for the channels of interest are suggested in Figure 5. A Green’s function specific to a z-directed magnetic current located in the appropriate interior equivalent model of Figure 5 is the fundamental source of the fields in the interior region (Appendix A). The component of the electric field that is tangential to the channel backing is
forced to be zero on the interior surface of the backing through the use of this specific Green's function. The integral equation enforcing continuity of tangential magnetic field in the slot is

$$
\mathcal{H}_{z,M}^{(1)}[2M_z; x, 0] - \mathcal{H}_{z,spec}^{(2)}[-M_z; x, 0] = -H_z^{SC}(x, 0), \quad x \in (x_1, x_2),
$$

(10)

where \(\mathcal{H}_{z,M}^{(1)}[M_z; x, 0]\) is given in (4), \(H_z^{SC}(x, 0)\) is given in (7), and \(\mathcal{H}_{z,spec}^{(2)}[-M_z; x, y]\) is the \(z\)-directed component of the magnetic field in the interior region. The terms \(\mathcal{H}_{z,M}^{(1)}[2M_z; x, 0]\) and \(H_z^{SC}(x, 0)\) are the same in the integral equations for each of the channels of Figure 1. Only \(\mathcal{H}_{z,spec}^{(2)}[-M_z; x, y]\) is different. \(\mathcal{H}_{z,spec}^{(2)}[-M_z; x, y]\) is necessarily different for the different channels because it is the contribution to the magnetic field in the slot due to the presence of the channel backing and it is formulated in such a way that that the boundary condition (29) is satisfied. \(\mathcal{H}_{z,spec}^{(2)}[-M_z; x, y]\) for the three specific channel backing structures are listed below.

**Hemicylindrical Channel [Figure 1a]**

\[
\mathcal{H}_{z,spec}^{(2)}[-M_z; x, y] = \mathcal{H}_{z,M}^{(3)}[-2M_z; x, y] - \frac{\kappa_2}{2\eta_2} \int_{x_1}^{x_2} M_z(x') \sum_{n=0}^{\infty} \epsilon_n \frac{H_n^{(2)}(k_2\rho') J_n(k_2\rho')}{J_n'(k_2\rho)} J_n(k_2\rho) \cos\theta(\phi - \phi') dx'.
\]

(11)

\[
\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases}
\]

(12)

\[
\rho = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)
\]

\[
\rho' = |x'|, \quad \phi' = \begin{cases} 0, & x' \geq 0 \\ \pi, & x' < 0 \end{cases}
\]

(13)

**Sector Channel [Figure 1b]**
\[ \mathcal{H}_{z, \text{spec}}^{(0)} [-M_z; x, y] = \frac{k_3 \pi}{2 \eta_2 \epsilon} \sum_{n=0}^{\infty} \epsilon_n \cos \left( k_\phi \phi \right) \int_{x_i}^{x_f} M_z (x') \cos \left( k_\phi \phi' \right) J_{k_{2\rho}} (k_2 \rho) H^{(2)}_{k_{2\rho}} (k_2 \rho') dx' \]

\[ - \frac{k_3 \pi}{2 \eta_2 \epsilon} \sum_{n=0}^{\infty} \epsilon_n \int_{x_i}^{x_f} M_z (x') \cos \left( k_\phi \phi' \right) \cos \left( k_\phi \phi \right) \frac{J_{k_{2\rho}} (k_2 \rho') H^{(2)}_{k_{2\rho}} (k_2 \rho)}{J_{k_{2\rho}'} (k_2 \rho')} J_{k_{2\rho}} (k_2 \rho) dx' \]

(14)

\[ k_\phi = \left( \frac{n \pi}{\alpha} \right), \quad \epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases} \quad \rho' = \begin{cases} \rho, & \rho < (>) \rho' \\ \rho', & \rho > (<) \rho' \end{cases} \]

(15)

Rectangular Channel [Figure 1c]

\[ \mathcal{H}_{z, \text{spec}}^{(3)} [-M_z; x, y] = -j \frac{\cos k_2 (y + d)}{\eta_2 h} \sin \frac{k_2 d}{\gamma_n} \int_{x_i}^{x_f} M_z (x') dx' \]

\[ + j \frac{2k_2}{\eta_2 h} \sum_{n=1}^{\infty} \frac{\cos \gamma_n (y + d)}{\gamma_n \sin \gamma_n d} \cos \frac{n \pi}{h} x \int_{x_i}^{x_f} M_z (x') \cos \frac{n \pi}{h} x' dx' \]

(16)

The term \( \gamma_n \) in equation (16) is given in (48). In (11) and (14), \( J_{\nu} (\bullet) \) is the \( \nu \) order Bessel function of the first kind. Equation (10) is solved for the magnetic current \( M_z (x) \) by typical method of moment numerical techniques. The magnetic current is expanded in a series of weighted pulses and the resulting equations are point matched to reduce the integral equation to a matrix equation.

Series Acceleration

Kummer’s method is applied to accelerate the convergence of the series in (16) [19]. One notes that the large-order form of the summation terms in (16) is

\[ \frac{\cos \gamma_n (y + d)}{\gamma_n \sin \gamma_n d} \cos \frac{n \pi}{h} x \cos \frac{n \pi}{h} x' \]

\[ \approx -\frac{h}{2 \pi n} e^{\frac{n \pi y}{h}} \left[ \cos \frac{n \pi}{h} (x + x') + \cos \frac{n \pi}{h} (x - x') \right] \quad \text{as} \ n \rightarrow \infty \]

(17)

and also that a closed form is known [20] for the summation.
\[ \sum_{n=1}^{\infty} \frac{a^n}{n} \cos n \theta = -\frac{1}{2} \ln \left( 1 - 2a \cos \theta + a^2 \right), \quad \theta = 2\pi, \quad a^2 < 1. \] (18)

The large-order form of the \( n^{th} \) term in (17) is subtracted from the original \( n^{th} \) term of the summation in (16), and the closed form of the summation in (18) is added to the expression. The resulting equation contains a summation that converges quickly and an additional closed-form term that is not a series:

\[ g_{z, \text{spec}}^{(3)}(-M_z; x, y) = \sum_{n=1}^{\infty} \frac{2k_z}{\eta_{zh}} \int_{x_l}^{x_h} M_n(x') dx' \]

\[ + j \frac{k_z}{\eta_{zh}} \int_{x_l}^{x_h} M_n(x') \left[ \sum_{n=1}^{\infty} \left( \frac{\cos \gamma_n (y + d)}{\sin \gamma_n d} \cos \frac{n\pi}{\eta_{zh}} x' \cos \frac{n\pi}{\eta_{zh}} x' \right. \right. \]

\[ \left. \left. + \frac{h}{2\pi} \cos \frac{n\pi}{\eta_{zh}} (x - x') \right] dx'. \] (19)

Equation (19) is equal to equation (16) because the subtracted summation and the added closed-form term are equal. However, the accelerated series of (19) converges in a few terms because the \( n^{th} \) term in the series is a difference which decays very rapidly.

Kummer’s method is applied also to accelerate the convergence of the first summation in (14).

One notes that the large-order form of the summation terms can be written as

\[ J_{kn} \left( k_{\phi} \right) H_{kn}^{(2)} \left( k_{\phi} \right) \cos \left( \frac{\pi \phi}{n} \right) \frac{\pi \phi}{n} \left( \frac{\beta}{\alpha} \right)^{\alpha n} \cos \left( \frac{\pi \phi}{\alpha} \right) \sim \frac{j\alpha}{n^2} \left( \frac{\beta}{\alpha} \right)^{\alpha n} \cos \left( \frac{\pi \phi}{\alpha} \right) \quad \text{as} \quad n \to \infty, \] (20)

which is of the same form as the left-hand side of (18). The expression for the magnetic field can now be rewritten as
\[ \mathcal{H}^{(2)}_{z,\text{spec}}[-M_z;x,y] = \frac{k_z\pi}{2\eta_2 \alpha} \int_{x_1}^{x_2} \frac{x'}{x_1} M_z(x') J_0(k_2\rho_x) H_0^{(2)}(k_2\rho_x) \, dx' + \frac{j k_2}{2\eta_2 \alpha} \sum_{n=1}^{\infty} \int_{x_1}^{x_2} M_z(x') \cos(k_2\phi) \left[ J_{k_n}(k_2\rho_x) H_{k_n}^{(2)}(k_2\rho_x) - \frac{j \alpha}{\pi m} \left( \frac{\rho_x}{\rho_\alpha} \right)^m \right] \, dx' \]

\[ - \frac{j k_2}{2\eta_2 \alpha} \int_{x_1}^{x_2} M_z(x') \ln \left( 1 - 2 \left( \frac{\rho_x}{\rho_\alpha} \right)^\alpha \cos \left( \frac{\pi}{\alpha} \phi \right) + \left( \frac{\rho_x}{\rho_\alpha} \right)^{2\alpha} \right) \, dx' \]

which converges much more quickly than (14).

**Fields from Equivalent Currents.**

The values for the pulse-expanded magnetic current determined from integral equation (10) are used to compute the fields near the structures of Figure 1. The total magnetic field in the exterior region is computed by evaluating (1) while the field in the interior region is computed from

\[ H_z^{(2)}(x,y) = \mathcal{H}^{(2)}_{z,\text{spec}}[-M_z;x,y] \]  

(22)

where \( \mathcal{H}^{(2)}_{z,\text{spec}}[-M_z;x,y] \) is given in (11), (14), and (16) for the case of a hemicylinder-, sector-, and rectangle-backed slot, respectively.

**Actual Currents from Fields and Equivalent Currents**

The actual currents present on the conducting surfaces of the structures in Figure 1 are determined from the values for the equivalent currents obtained from the integral equation methods discussed in the previous sections. From knowledge of the equivalent currents found as solutions of the integral equations, the total magnetic field is computed along the contours originally occupied by conductors, and this magnetic field is related to the actual currents present on the original structure.
The x-directed electric current present on the surface of the conducting plane in the exterior region is

\[ J_z^{(1)} (x) = (\hat{y} \times \hat{z}) H_z^{(1)} (x, 0) = \hat{x} H_z^{(1)} (x, 0), \quad x \notin (x_1, x_2) \]  \hspace{1cm} (23)

where \( H_z^{(1)} (x, y) \) is given in (1). Similarly, the current on the baffle is

\[ J_z^{(2)} (x) = (\hat{y} \times \hat{z}) H_z^{(2)} (x, 0) = -\hat{x} H_z^{(2)} (x, 0), \quad x \in \text{baffle} \]  \hspace{1cm} (24)

where \( H_z^{(2)} (x, y) \) is given by

\[ H_z^{(2)} (x, y) = \gamma_{s,M} [-2M_s; x, y] + \gamma_{s,1} \left[ J_e; x, y \right] \]  \hspace{1cm} (25)

for the coupled integral equation method and by (22) for the specific Green’s function method, where the baffle is defined as the surface of the conducting plane in the interior region bounded by the channel backing. The current on the interior surface of the channel backing is determined by evaluating

\[ J_e (x, y) = (\hat{n} \times \hat{z}) \gamma_{s,\text{spec}} [-M_s; x, y] = \hat{y} \gamma_{s,\text{spec}} [-M_s; x, y], \quad (x, y) \in C \]  \hspace{1cm} (26)

for the specific Green’s function method, where \( C \) is the contour of the original channel backing in Figure 1. In the use of the coupled integral equation method no further evaluation is necessary to obtain the actual currents on the interior surface of the channel backing. The equivalent electric current on the channel contour in Figure 4b is identical to the actual current on the original channel backing, because the field is zero in the lower half-space outside of the channel and the total magnetic field must be discontinuous by the value of the equivalent electric current, i.e.,

\[ J_e^{\text{equiv}} = (\hat{n} \times \hat{z}) \left[ H_z^{(2)} (x, y) \big|_{(x,y) \in C^*} - H_z^{(2)} (x, y) \big|_{(x,y) \in C^*} \right] = \hat{y} H_z^{(2)} (x, y) \big|_{(x,y) \in C^*} = J_e^{\text{actual}} \]  \hspace{1cm} (27)
where $J_{i}^{\text{actual}}$ is the actual current on the interior surface of the channel, $J_{i}^{\text{equiv}}$ is the equivalent electric current obtained from the coupled integral equations, $C^{-}$ is the contour on the inside surface of the channel, and $C^{+}$ is on the outside surface of the channel.

One notes that in obtaining the actual currents the magnetic field is evaluated at locations for which there are no field discontinuities in the equivalent models.

### 3. Data

The equivalent magnetic current on the shorted aperture is determined in each structure from the solution of either coupled integral equations or a channel-specific single integral equation. The fields and currents everywhere are then found from the solutions of the integral equations. The data from the two independent integral equation methods are presented in Figures 6-10 and are shown to agree well in all cases. The coupled integral equation method is also used to compute the total fields near and the currents induced on a semicircular-backed channel which is filled with a material isoreflective to that of the exterior region. Data obtained from the coupled integral equation method for this case are presented in Figures 11-12 along with those of a method involving the evaluation of analytical expressions containing summations of Mathieu functions ([12] [13]).

### 4. Conclusions

The field near the structures of Figure 1 is computed through two independent integral equation methods: a coupled integral equation method that is valid for an arbitrarily-shaped channel-backed slot and a single integral equation with Green's function specific to the shape of the channel backing. Data are presented for the equivalent magnetic currents on the shorted aperture in the equivalent models, the total field near the structures, and the actual currents present on the conductors of the channel-backed slot. Data from the two methods are shown to agree well in all
cases considered. The integral equation techniques provide two distinct methods for determining the penetration through an infinite slot backed by a channel filled, possibly, with material different from the material in the half-space in front of the ground plane.
Appendix A

Green’s Function for Magnetic Line Source in Various Channels

In this appendix, the three Green’s functions needed in the single integral equation are described. In each case, the Green’s function is the electric vector potential due to a z-directed magnetic line current of unity strength located at \( \rho = \rho' \) inside the channel backing as suggested in Figure 13. This magnetic vector potential \( f \hat{z} \) satisfies the differential equation,

\[
(\nabla_i^2 + k^2) f = -\varepsilon \delta(\rho - \rho'),
\]

(28)

where \( \nabla_i^2 \) is the transverse-to-z Laplacian in the coordinate system compatible with the channel shape. The electric field in the channel must satisfy the boundary condition

\[ \mathbf{E} \times \hat{n} = 0 \quad \text{on } C \]

(29)

where \( C \) is the contour of the conducting channel backing and \( \hat{n} \) is the unit vector normal to this surface. The boundary conditions for the three channel backings are listed in Table 1.

Table 1: Boundary Conditions for Channel Backing Shapes

<table>
<thead>
<tr>
<th>Backing</th>
<th>Hemicylinder</th>
<th>Sector</th>
<th>Rectangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditions</td>
<td>( E_o (\rho, \phi) \big</td>
<td>_{\rho=\alpha} = 0 ) (30)</td>
<td>( E_{\phi} (\rho, \phi) \big</td>
</tr>
<tr>
<td></td>
<td>( E_{\rho} (\rho, \phi) \big</td>
<td>_{\phi=\alpha} = 0 ) (31)</td>
<td>( E_{\phi} (\rho, \phi) \big</td>
</tr>
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</table>

Green’s Function for Hemicylindrical Channel. The Green’s function for the hemicylindrical channel backing of Figure 13a is the solution to (28) in circular cylindrical coordinates subject to boundary conditions (30) and (31). Image theory is invoked to create a model that produces the same fields as the original configuration for \( y < 0 \) (Figure 14). Boundary condition (31) is enforced through symmetry in the resulting model. We represent the Green’s function as the sum of a particular and a homogeneous solution of (28) with the particular solution taken to be
\[ f^p(\rho, \phi) = \frac{\varepsilon_2}{4j} \left[ H_0^{(2)}(k_2 R) + H_0^{(2)}(k_2 R') \right] \]
\[ = \frac{\varepsilon_2}{2j} \sum_{n=-\infty}^{\infty} J_n(k_2 \rho) H_n^{(2)}(k_2 \rho') e^{im\phi} \cos n\phi' \]  

where \(R = \sqrt{\rho^2 + \rho'^2 - 2 \rho \rho' \cos(\phi - \phi')}\), \(R' = \sqrt{\rho^2 + \rho'^2 - 2 \rho \rho' \cos(\phi + \phi')}\), and

\[ \rho_\text{<} = \begin{cases} \rho & \rho < \rho' \\ \rho' & \rho > \rho' \end{cases}, \quad \rho_\text{>} = \begin{cases} \rho' & \rho < \rho' \\ \rho & \rho > \rho' \end{cases} \]  

\(f^p\) of (36) is the potential due to two magnetic line currents of unity strength located at \((\rho', \phi')\) and \((\rho', -\phi')\) in open space [18]. The homogeneous solution which is bounded at the origin of coordinates is of the form

\[ f^h(\rho, \phi) = -\frac{\varepsilon_2}{2j} \sum_{n=-\infty}^{\infty} C_n J_n(k_2 \rho) e^{im\phi} \cos n\phi'. \]  

The constant \(C_n\) is determined by computing \(E_\phi\) from the total potential \(f^p(\rho, \phi) + f^h(\rho, \phi)\) and enforcing the boundary condition (30) on the cylinder surface. With \(C_n\) available and substituted into (38), one expresses the total vector potential as

\[ f(\rho, \phi) = \frac{\varepsilon_2}{4j} \left[ H_0^{(2)}(k_2 R) + H_0^{(2)}(k_2 R') \right] \]
\[ - \frac{\varepsilon_2}{2j} \sum_{n=-\infty}^{\infty} \frac{H_n^{(2)}(k_2 \rho) J_n(k_2 \rho') J_n(k_2 \rho) e^{im\phi} \cos n\phi'}. \]  

For computational purposes, the first term on the right side of (39) is more convenient than its equivalent in series form in (36), which is introduced only to facilitate evaluation of the constant \(C_n\).
Green's Function for Sectorial Channel. Primary and homogeneous solutions of (28) are constructed in forms that satisfy the boundary condition (33) and then (32) is enforced. The primary portion of the electric vector potential is a solution of

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} f^p (\rho, \phi) \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} f^p (\rho, \phi) + k_2^2 f^p (\rho, \phi) = 0, \quad \rho \neq \rho'. \quad (40)$$

Solving (40) in cylindrical coordinates subject to condition (33), one obtains the primary solution in the form

$$f^p (\rho, \phi) = \sum_{n=0}^{\infty} b_n \cos \left( k_{\rho_n} \rho \right) J_{k_{\rho_n}} \left( k_{\rho_n} \rho \right) H_{k_{\rho_n}}^{(2)} \left( k_{\rho_n} \rho \right), \quad \rho \neq \rho' \quad (41)$$

with

$$k_{\rho_n} = \left( \frac{n \pi}{\alpha} \right). \quad (42)$$

The value of $k_{\rho_n}$ in (42) causes the electric field computed from (41) to satisfy (33). The form chosen for the primary solution in (41) forces $H_z$ to be continuous in $\rho$ at $\rho = \rho'$. The value of $b_n$ is determined from the jump discontinuity that $E_\phi$ must exhibit at a surface magnetic current, which in terms of $f^p (\rho, \phi)$ is

$$-\frac{1}{\varepsilon_2} \left[ \frac{\partial}{\partial \rho} f^p (\rho, \phi) \right]_{\rho=\rho'} - \left[ \frac{\partial}{\partial \rho} f^p (\rho, \phi) \right]_{\rho=\rho'} = \frac{1}{\rho'} \delta (\phi - \phi'). \quad (43)$$

Expansion of the right-hand side of (43) in a Fourier series allows one to determine $b_n$:

$$b_n = \begin{cases} \frac{j \varepsilon_2 \pi}{2 \alpha}, & n = 0 \\ \frac{j \varepsilon_2 \pi}{\alpha} \cos \left( k_{\rho_n} \rho' \right), & n \neq 0 \end{cases} \quad (44)$$

The homogeneous solution is constructed in a form satisfying (33) but embodying the usual arbitrary constant. The total $\phi$-directed electric field is written as
\[ E_\phi (\rho, \phi) = \frac{1}{\varepsilon_2} \frac{\partial}{\partial \rho} f^p (\rho, \phi) + \frac{1}{\varepsilon_2} \frac{\partial}{\partial \rho} f^h (\rho, \phi) \]  

(45)

and the arbitrary constant in \( f^h (\rho, \phi) \) is found by subjecting (45) to boundary condition (32), resulting in

\[ f^h (\rho, \phi) = -\sum_{n=0}^{\infty} b_n \cos (k_n \rho) \left( \frac{J_{k_n} (k_n \rho')}{J_{k_n} (k_n \rho)} \right) \frac{H_{k_n}^{(2)'} (k_n \rho)}{J_{k_n}^{(2)'} (k_n \rho)}. \]  

(46)

The total electric vector potential is \( f (\rho, \phi) = f^p (\rho, \phi) + f^h (\rho, \phi) \).

\textit{Green's Function for Rectangular Channel.} The Green's function for the rectangular channel backing of Figure 13c is the solution of (28) in rectangular coordinates subject to boundary conditions (34) and (35). The Green's function is expressed as the sum of a particular and homogeneous solution of (28) with the particular solution given by

\[ f^p (x, y) = -j \frac{\varepsilon_2}{2 k_2 h} e^{-j k_2 |y-y'|} - j \frac{\varepsilon_2}{h} \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \cos \left( \frac{n \pi}{h} x \right) \cos \left( \frac{n \pi}{h} x' \right) e^{-ir_n |y-y'|} \]  

(47)

where

\[ \gamma_n = \begin{cases} -j \sqrt{\left( \frac{n \pi}{h} \right)^2 - k^2}, & k^2 < \left( \frac{n \pi}{h} \right)^2 \\ \sqrt{k^2 - \left( \frac{n \pi}{h} \right)^2}, & k^2 > \left( \frac{n \pi}{h} \right)^2 \end{cases}. \]  

(48)

Equation (47) is the electric vector potential due to a \( z \)-directed magnetic line source located at \( (x', y') \) between two infinite conducting plates located at \( x = 0 \) and \( x = h \) ([21], [22], [23]).

The homogeneous solution is postulated such that the \( x \) variation of the solution is identical to that of (47) and reflected waves in the \( \pm y \) directions are represented, which allows one to enforce boundary condition (35). The general form of the homogeneous solution is
\[ f^h(x, y) = -j \frac{\varepsilon_2}{2k_y h} \left[ C_{01} e^{\beta y} + C_{02} e^{-\beta y} \right] \]

\[ - j \frac{\varepsilon_2}{h} \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \cos \left( \frac{n\pi}{h} x \right) \cos \left( \frac{n\pi}{h} x' \right) \left[ C_{01} e^{\gamma_n y} + C_{02} e^{-\gamma_n y} \right] \quad (49) \]

The coefficients \( C_{01} \), \( C_{02} \), \( C_{01} \), and \( C_{02} \) of (49) are determined by enforcing boundary condition (35) on the \( x \)-directed component of the total electric field, namely

\[ E_x(x, y) = -\frac{1}{\varepsilon_2} \frac{\partial}{\partial y} f^p(x, y) - \frac{1}{\varepsilon_2} \frac{\partial}{\partial y} f^h(x, y) = 0 \quad , \quad y = 0, -d \quad (50) \]

The resulting expression for the homogeneous solution is

\[ f^h(x, y) = -\frac{\varepsilon_2}{2k_y h} \left[ \frac{\cos k_y (y + d)}{\sin k_y d} e^{\beta y} + \frac{\cos k_y y'}{\sin k_y d'} e^{-\beta (y + d)} \right] \]

\[ - \frac{\varepsilon_2}{h} \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \cos \left( \frac{n\pi}{h} x \right) \cos \left( \frac{n\pi}{h} x' \right) \left[ \frac{\cos \gamma_n (y + d)}{\sin \gamma_n d} e^{\gamma_n y} + \frac{\cos \gamma_n y'}{\sin \gamma_n d'} e^{-\gamma_n y + d} \right] \quad (51) \]

The total electric vector potential is \( f(x, y) = f^p(x, y) + f^h(x, y) \). 

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8. Figures

Figure 1: TE-Excited Channel-Backed Slots; a) Hemicylinder-Backed Slot; b) Sector-Backed Slot; c) Rectangle-Backed Slot

Figure 2: TE-Excited Slot with Arbitrarily-Shaped Backing

Figure 3: Channel-Backed Slot with Shorted Aperture and Impressed Magnetic Current
Figure 4: a) Exterior Region Equivalent Model; b) Interior Region Equivalent Model for Coupled Integral Equations Method

Figure 5: Interior Region Equivalent Model for TE-Excited Channel-Backed Slots; a) Hemicylinder-Backed Slot; b) Sector-Backed Slot; c) Rectangle-Backed Slot
Figure 6: Magnetic Surface Current on Shorted Aperture of a) Hemicylinder-Backed Slot Excited by Normally-Incident TE-Polarized Plane Wave of Unity Strength; $a=0.9\; m$, $x_1=0.2\; m$, $x_2=0.5\; m$; $(\mu_1, \varepsilon_1) = (\mu_2, \varepsilon_2) = (\mu_0, \varepsilon_0)$; b) Rectangle-Backed Slot Excited by Magnetic Line Source of Unity Strength Located at $(x_s, y_s) = (1\; m, 1\; m)$; $h=d=0.7\; m$, $x_1=0.1\; m$, $x_2=0.3\; m$; $(\mu_1, \varepsilon_1) = (\mu_2, \varepsilon_2) = (\mu_0, \varepsilon_0)$; c) Sector-Backed Slot Excited by Magnetic Line Source of Unity Strength Located at $(x_s, y_s) = (-1\; m, 1\; m)$; $a=1.3\; m$, $\alpha=45^\circ$, $x_1=0.2\; m$, $x_2=0.6\; m$; $(\mu_1, \varepsilon_1) = (\mu_0, \varepsilon_0)$, $(\mu_2, \varepsilon_2) = (2\mu_0, 2\varepsilon_0)$; Freq. = 300 MHz
Figure 7: a) Total Magnetic Field Along X=0.4m for Hemicylinder-Backed Slot Excited by Normally-Incident TE-Polarized Plane Wave of Unity Strength ; a=0.9 m ; x1=0.2m, x2=0.5m ; (μ₁,ε₁) = (μ₂,ε₂) = (μ₀,ε₀) ; b) Total Magnetic Field Along X=0.1m for Rectangle-Backed Slot Excited by Magnetic Line Source of Unity Strength Located at (xₛ,yₛ)=(1 m,1 m) ; h=d=0.7 m ; x₁=0.1 m, x₂=0.3 m ; (μ₁,ε₁) = (μ₂,ε₂) = (μ₀,ε₀) ; c) Total Magnetic Field Along X=0.5m for Sector-Backed Slot Excited by Magnetic Line Source of Unity Strength Located at (xₛ,yₛ)=(-1 m,1 m) ; a=1.3 m ; α=45° ; x₁=0.2 m, x₂=0.6 m ; (μ₁,ε₁) = (μ₀,ε₀) , (μ₂,ε₂) = (2μ₀,2ε₀) ; Freq. = 300 MHz
Figure 8: Electric Current on Surfaces of Hemicylinder-Backed Slot Excited by Normally-Incident TE-Polarized Plane Wave of Unity Strength; \( a = 0.9 \, \text{m} \); \( x_1 = 0.2 \, \text{m} \); \( x_2 = 0.5 \, \text{m} \); \((\mu_1, \varepsilon_1) = (\mu_2, \varepsilon_2) = (\mu_0, \varepsilon_0)\)

Figure 9: Electric Current on Surfaces of Rectangle-Backed Slot Excited by Magnetic Line Source of Unity Strength Located at \((x_s, y_s) = (1 \, \text{m}, 1 \, \text{m})\); \( h = d = 0.7 \, \text{m} \); \( x_1 = 0.1 \, \text{m} \); \( x_2 = 0.3 \, \text{m} \); \((\mu_1, \varepsilon_1) = (\mu_2, \varepsilon_2) = (\mu_0, \varepsilon_0)\); Freq. = 300 MHz
Figure 10: Electric Current on Surfaces of Sector-Backed Slot Excited by Magnetic Line Source of Unity Strength Located at \((x_0,y_0)=(-1\, \text{m}, 1\, \text{m})\); \(a=1.3\, \text{m}\); \(\alpha=45^{\circ}\); \(x_1=0.2\, \text{m}\), \(x_2=0.6\, \text{m}\); \((\mu_1,\varepsilon_1) = (\mu_0,\varepsilon_0), (\mu_2,\varepsilon_2) = (2\mu_0,2\varepsilon_0)\); Freq. = 300 MHz

\[ H_z \text{ Along Y-Axis for Plane Wave at } 45^\circ \text{ Incidence} \]
\[ Z_1 = 120\pi, Z_2 = 240\pi \]

Figure 11: Magnetic Field Along Y-Axis for Semielliptical-Backed Slot Excited by Plane Wave of Unity Strength and Incidence \(\theta = 45^{\circ}\); \((\mu_1,\varepsilon_1) = (\mu_0,\varepsilon_0), (\mu_2,\varepsilon_2) = (2\mu_0,0.5\varepsilon_0)\); \(d=2\); \(b=1.732\)
Freq. = 214.7 MHz
Figure 12: Currents on Surfaces of Semielliptical-Backed Slot Excited by Plane Wave of Unity Strength and Incidence $\theta = 45^\circ$; $(\mu_1, \varepsilon_1) = (\mu_0, \varepsilon_0)$, $(\mu_2, \varepsilon_2) = (2\mu_0, 0.5\varepsilon_0)$; $d = 2$; $b = 1.732$ Freq. = 214.7 MHz

Figure 13: Z-Directed Magnetic Line Source of Unity Strength ($K = 1$) Located at $p'$ in a) Conducting Hemicylindrical Channel b) Conducting Sectorial Channel c) Conducting Rectangular Channel
Figure 14: Equivalent Model for Line Source of Unity Strength \((K = 1)\) in Conducting Hemicylindrical Channel